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In [3] Knaster and Reichbach proved that any homeomorphism defined on a closed subset P of the Cantor set C can be extended to a homeomorphism of the Cantor set onto itself. It was proven in [1] and [2] that if P is closed and nowhere dense, then the extended homeomorphism has the following property: an orbit of a point in C-P is dense in C. In fact, there is a dense G_σ of such points. We prove here that in a special case, when the given homeomorphism is an identity on P, an orbit of any point in C-P is dense in C-P.

By C we denote the Cantor set in the closed unit interval [0,1] done by the usual ternary construction.

Let $\mathcal{B}_n$ denote the family of all sets $[\frac{k-1}{3^n}, \frac{k}{3^n}] \cap C$ that consist of more than two points, where $k = 1,2,\cdots,3^n$. Hence $\mathcal{B}_n$ is a family of 2^n closed-open and disjoint subsets of C and $\mathcal{B} = \cup\{\mathcal{B}_n : n = 1,2,3,\cdots\}$ is a basis of C.

If f is a homeomorphism on C then, for any integer n, f^n is defined by

$$f^0(x) = x$$

$$f^{n+1}(x) = f(f^n(x))$$

where $x \in C$.

Theorem. If D *is a closed subset of the Cantor set* C *then there exists a homeomorphism* f *from* C *onto itself such* *that* $f|_D = id|_D$ *and for every point* $c \in (C - D)$ *the set* $\{f^n(c)\colon n$ *is an integer}* *is dense in* $C - D$.

Proof. If $D = \emptyset$ or $D = C$ then the theorem is true. Suppose then that D is a proper non-empty subset of C. Let $\mathcal{U}$ be a family of pairwise disjoint elements of $\mathcal{B}$ such that $\cup\mathcal{U} = C - D$.

For every $B \in \mathcal{B}$ let $B(1)$ and $B(2)$ denote elements of $\mathcal{B}$ such that $\mathrm{diam}B(1) = \mathrm{diam}B(2) = 3^{-1}\cdot\mathrm{diam}B$, $B(1) \cup B(2) = B$, and if $x_1 \in B(1)$ and $x_2 \in B(2)$ then $x_1 < x_2$.

For every positive integer n we define a function f_n and a family $\mathcal{A}_n$ such that the following conditions are fulfilled:

(i) If $A \in \mathcal{A}_n$ then $\mathrm{diam}A \leq 3^{-n}$.

(ii) $\cup\mathcal{A}_{n-1} \subseteq \cup\mathcal{A}_n$.

(iii) If $U \in \mathcal{U}$ and $\mathrm{diam}U \geq 3^{-n}$ then $U \subseteq \cup\mathcal{A}_n$.

(iv) Let $B \in \mathcal{B}_n$. If $B \cap D = \emptyset$ then $B \subseteq \cup\mathcal{A}_n$. If $B - (\cup\mathcal{A}_{n-1} \cup D) \neq \emptyset$ then there is $U \in \mathcal{U}$ such that $U \subseteq B - (\cup\mathcal{A}_{n-1} \cup D)$ and $U(1), U(2) \in \mathcal{A}_n$.

(v) If $A \in \mathcal{A}_n$ then $\mathcal{A}_n = \{f_n^i(A)\colon i = 1,2,\cdots,m(n)\}$, where $m(n)$ is a number of elements in $\mathcal{A}_n$.

(vi) For any $A \in \mathcal{A}_n$ a restriction $f_n|_A$ is an increasing and linear function from A onto $f_n(A)$.

(vii) $f_n|_{C-\cup\mathcal{A}_n} = id|_{C-\cup\mathcal{A}_n}$.

(viii) $f_n(A) = f_{n-1}(A)$ for $A \in \mathcal{A}_{n-2}$.

(ix) For every $x \in C$ we have $|f_n(x) - f_{n-1}(x)| \leq 3^{1-n}$.

(x) If $B \in \mathcal{B}_n$ and $B \cap (\cup\mathcal{A}_n - \cup\mathcal{A}_{n-1}) \neq \emptyset$ then there are $A_1, A_2 \in \mathcal{A}_n$ such that $A_1, A_2 \subseteq B \cap (\cup\mathcal{A}_n - \cup\mathcal{A}_{n-1})$ and $f_n(A_1) = A_2$.

Step 1. The family $\mathcal{B}_1$ consists of two sets, say B_1 and B_2. For i = 1,2 we put $\mathcal{A}_1(i) = \{U(s): s = 1,2$ and $U \in \mathcal{U}$ and $U \subseteq B_i\}$, if $B_i \cap D = \emptyset$. If $B_i \cap D \neq \emptyset$ and $B_i - D \neq \emptyset$ then we put $\mathcal{A}_1(i) = \{U(1),U(2)\}$ where $U \in \mathcal{U}$ and $U \subseteq B_i - D$. We put $\mathcal{A}_1 = \mathcal{A}_1(1) \cup \mathcal{A}_1(2)$. Families $\mathcal{A}_1(1)$ and $\mathcal{A}_1(2)$ are finite, say $\mathcal{A}_1(1) = \{A_1,\cdots,A_{m(1)}\}$ and $\mathcal{A}_1(2) = \{A_1^*,\cdots,A_{m(2)}^*\}$. We define f_1 so that

- $f_1|_{A_j}$ is a linear and increasing function from A_j onto A_{j+1} for $j = 1,\cdots,m(1)-1$
- $f_1|_{A_m(1)}$ is a linear and increasing function from $A_{m(1)}$ onto A_1^*
- $f_1|_{A_j^*}$ is a linear and increasing function from A_j^* onto A_{j+1}^* for $j = 1,\cdots,m(2)$
- $f_1|_{A_{m(2)}^*}$ is a linear and increasing function from $A_{m(2)}^*$ onto A_1
- $f_1|_{C-\cup\mathcal{A}_1} = id|_{C-\cup\mathcal{A}_1}$.

It is easy to see that conditions (i)-(x) are fulfilled.

Step n+1. Suppose that we have defined families $\mathcal{A}_k$ and functions f_k for $k = 1,2,\cdots,n$. Let the elements of $\mathcal{B}_{n+1}$ be denoted by B_j, $j = 1,2,\cdots,2^{n+1}$, in such a way that $(B_{2i} \cup B_{2i-1}) \in \mathcal{B}_n$ for $i = 1,2,\cdots,2^n$. Let $\mathcal{A}_{n+1} = \{A(s): s = 1,2$ and $A \in \mathcal{A}_n\}$. Let $\mathcal{A}_{n+1}(j) = \{U(s): s = 1,2$ and $U \in \mathcal{U}$ and $U \subseteq B_j - \cup\mathcal{A}_n\}$ if $B_j \cap D = \emptyset$. If $B_j \cap D \neq \emptyset$ and $B_j - (D \cup \cup\mathcal{A}_n) \neq \emptyset$ then we put $\mathcal{A}_{n+1}(j) = \{U(1),U(2)\}$ for some $U \in \mathcal{U}$ such that $U \subseteq B_j - (D \cup \cup\mathcal{A}_n)$.

Let $\mathcal{A}_{n+1} = \cup\{\mathcal{A}_{n+1}(j): j = 0,1,\cdots,2^{n+1}\}$. Conditions (i)-(iv) are satisfied by $\mathcal{A}_{n+1}$.

Let A be a fixed element of $\mathcal{A}_n$. Define function g_o by

- $g_o|_{C-A} = f_n|_{C-A}$

- $g_o|_{A(1)}$ is an increasing and linear function from A(1) onto $f_n(A(2))$

- $g_o|_{A(2)}$ is an increasing and linear function from A(2) onto $f_{n(A(1))}$.

Conditions (v)-(x) are satisfied by g_o and $\mathcal{A}_{n+1}(0)$. If $\mathcal{A}_{n+1} = \mathcal{A}_{n+1}(0)$ then we put $f_n = g_o$.

Suppose that $\mathcal{A}_{n+1} \neq \mathcal{A}_{n+1}(0)$. Because $(B_{2i-1} \cup B_{2i}) \in \mathcal{B}_n$ for $i = 1,2,\cdots,2^n$, then diameter of $\cup(\mathcal{A}_{n+1}(2i-1) \cup \mathcal{A}_{n+1}(2i))$ is less than or equal to 3^{-n}. Consider $\mathcal{A}_{n+1}(1) \cup \mathcal{A}_{n+1}(2)$. If it is an empty set then we put $g_1 = g_o$. Otherwise it is finite. Suppose that both $\mathcal{A}_{n+1}(1)$ and $\mathcal{A}_{n+1}(2)$ are not empty and put $\mathcal{A}_{n+1}(1) = \{A_1,\cdots,A_{m(1)}\}$ and $\mathcal{A}_{n+1}(2) = \{A_1^*,\cdots,A_{m(2)}^*\}$. Because $B_1 \cup B_2$ is an element of $\mathcal{B}_n$ and $(B_1 \cup B_2) - (\cup\mathcal{A}_{n-1} \cup D) \neq \emptyset$ then, by (iv), (x), and the definitions of $\mathcal{A}_n(0)$ and g_o there are $E_1,E_2 \in \mathcal{A}_n(0)$ such that $E_1,E_2 \subseteq (B_1 \cup B_2) - (\cup\mathcal{A}_n - \cup\mathcal{A}_{n-1})$ and $g_o(E_1) = E_2$. We define g_1 from C onto itself as follows:

- $g_1|_{C-(E_1 \cup \cup\mathcal{A}_{n+1}(1) \cup \cup\mathcal{A}_{n+1}(2))} =$

$g_o|_{C-(E_1 \cup \cup\mathcal{A}_{n+1}(1) \cup \cup\mathcal{A}_{n+1}(2))}$

- $g_1|_{E_1}$ is a linear and increasing function from E_1 onto A_1

- $g_1|_{A_r}$ is a linear and increasing function from A_r onto A_{r+1} for $r = 1,2,\cdots,m(1)-1$

- $g_1|_{A_{m(1)}}$ is a linear and increasing function from $A_{m(1)}$ onto A_1^*
- $g_1|_{A_r^*}$ is a linear and increasing function from A_r^* onto A_{r+1}^* for $r = 1,2,\cdots,m(2)-1$
- $g_1|_{A_{m(2)}^*}$ is a linear and increasing function from $A_{m(2)}^*$ onto E_2.

If one of the families $\mathcal{A}_{n+1}(1)$, $\mathcal{A}_{n+1}(2)$ is empty then some obvious modifications of the preceding process are required. In any case it is easy to see that the family $\mathcal{A}_{n+1}(0) \cup \mathcal{A}_{n+1}(1) \cup \mathcal{A}_{n+1}(2)$ and the function g_1 satisfy (i)-(x). We repeat this procedure for families $\mathcal{A}_{n+1}(2i-1) \cup \mathcal{A}_{n+1}(2i)$, where $i = 2,3,\cdots,2^n$.

If we put $f_{n+1} = g_{2^n}$ then (i)-(x) are fulfilled for such a function and the family $\mathcal{A}_{n+1}$.

Conditions (i)-(x) imply that $f = \lim_{n\to\infty} f_n$ is a homeomorphism on C that is an identity on D and $\{f^k(c)$: k is an integer$\}$ is dense in C - D for any point c in C - D.

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