TOPOLOGY PROCEEDINGS
Volume 43, 2014
Pages 29–36

http://topology.auburn.edu/tp/

# On compact sets in $C_b(X)$

by

Juan Carlos Ferrando

Electronically published on April 29, 2013



**Topology Proceedings**

| | |
|---|---|
| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings<br>Department of Mathematics & Statistics<br>Auburn University, Alabama 36849, USA |
| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 43 (2014)
Pages 29-36
E-Published on April 29, 2013

# ON COMPACT SETS IN $C_b(X)$

JUAN CARLOS FERRANDO

Abstract. Let $X$ be a Tychonoff space and $C_p(X)$ and $C_b(X)$ denote the space $C(X)$ of all real-valued continuous functions on $X$ provided with the pointwise convergence and the compact-bounded topology, respectively. Let us call an unbounded subspace $\Sigma$ of $\mathbb{N}^{\mathbb{N}}$ $\sigma$-complete if each bounded sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\Sigma$ verifies that $\sup_{n\in\mathbb{N}}\alpha_n \in \Sigma$, and let us call $\sigma$-complete every family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of $X$ such that $\Sigma$ is $\sigma$-complete and $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$. We show that if there is a dense subspace of $X$ covered by a $\sigma$-complete family consisting of bounded sets then $C_p(X)$ is angelic. This is used to prove our main theorem, which asserts that if $X$ has a dense subspace $Y$ covered by a $\sigma$-complete family consisting of $Y$-bounded sets, then $C_b(X)$ is angelic and every compact set in $C_b(X)$ is metrizable.

## 1. Preliminaries

Let us start by recalling that a subset $A$ of a topological space $X$ is called (functionally) *bounded* [1, Chapter 0] if $f(A)$ is a bounded set in $\mathbb{R}$ for every real-valued continuous function $f$ on $X$. On the other hand, a subset $B$ of a topological vector space $E$ is (linearly) *bounded* [9, Chapter 3] if $B$ is absorbed by every neighborhood of the origin. In what follows, unless otherwise stated, $X$ will be a Hausdorff completely regular space and $C_p(X)$, $C_c(X)$ and $C_b(X)$ will denote the space $C(X)$ of all real-valued continuous functions defined on $X$ equipped with the pointwise convergence topology, the compact-open and the compact-bounded topology, respectively. We denote by $L(X)$ the topological dual of $C_p(X)$ and by $L_p(X)$ the linear space $L(X)$ endowed with the weak* topology.

2010 *Mathematics Subject Classification.* 54D30, 54E15, 46A03.
*Key words and phrases.* Compact set, pointwise convergence topology, compact-bounded topology.